PERGAMON

International Journal of Heat and Mass Transfer 43 (2000) 2565–2571

International Journal of HEAT and MASS TRANSFER

www.elsevier.com/locate/ijhmt

Vertical free convection in a porous medium with variable permeability effects

D.A.S. Rees [a,*], I. Pop [b]

[a] Department of Mechanical Engineering, University of Bath, Bath BA2 7AY, UK
[b] Faculty of Mathematics, University of Cluj, R-3400 Cluj, CP 253, Romania

Received 10 September 1999

Abstract

When a porous medium is bounded by an impermeable surface it is well known that the porosity and hence the permeability increases near that surface. In this paper, we examine how this phenomenon affects the flow and heat transfer from a uniform temperature heated surface. In particular, we assume that the region of varying permeability has constant thickness, and we present detailed numerical and asymptotic solutions for the resulting nonsimilar flow. Near the leading edge the flow is enhanced and the rate of heat transfer is much higher than in the uniform permeability case. Further downstream the region of varying permeability is embedded well within the boundary layer, and in this case the flow and heat transfer is only slightly different from that of the uniform case. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Free convection; Porous media; Boundary layer; Variable permeability

1. Introduction

The large number of published papers on convective heat transfer and fluid flow through porous media demonstrates clearly that this area of fluid mechanics has been studied extensively during the last three decades. This research activity has accelerated because of a broad range of applications in contemporary technology, such as geophysics, thermal insulation engineering, packed-sphere beds, grain storage, heat exchangers, the cooling of electronic components, groundwater hydrology, nuclear waste repositories and chemical catalytic reactors. Recent books by Nield and Bejan [1] and Ingham and Pop [2] present a comprehensive account of the available information on these flows, and, in particular, stress the importance of the many extensions to Darcy's law which are needed in various applications; these include inertia (form drag), boundary (Brinkman), thermal dispersion and local thermal nonequilibrium effects. However, most studies neglect these extensions and concentrate mainly on flows when Darcy's law and the Boussinesq approximation are valid, and when thermal conduction and advection are governed by an equation identical in form to that applying for a viscous clear fluid.

In high porosity media boundary and inertia effects become important as the fluid velocities are quite high and the rate of heat transfer from the heated surface is evaluated in that near-wall region where boundary effects are most significant. The influence of boundary and inertia effects on forced

* Corresponding author. Tel.: +44-1225-826775; fax: +44-1225-826928.

E-mail address: ensdasr@bath.ac.uk (D.A.S. Rees).

0017-9310/00/$ - see front matter
PII: S0017-9310(99)00316-6

Nomenclature

a_0	constant
A	dimensionless permeability variation
d	lengthscale associated with the permeability variation
f, F	reduced streamfunction
g, G	reduced temperature
K	permeability
p	pressure
Q	rate of heat transfer
Ra	Darcy–Rayleigh number
u	flux velocity in the x-direction
v	flux velocity in the y-direction
x	coordinate along the heated surface
y	coordinate perpendicular to the heated surface

Greek symbols

γ	permeability ratio
ψ	streamfunction
ξ	scaled streamwise coordinate
η	pseudo-similarity variable
θ	temperature
λ	eigenvalue

Superscripts and subscripts

–	boundary layer scalings
*	dimensional
∞	ambient conditions
w	at heated surface
x	derivative with respect to x
y	derivative with respect to y
$'$	derivative with respect to η or $\hat{\eta}$
ˆ	scaled variables near the leading edge
0, 1, 2	terms in the $\xi \gg 1$ analysis
˜	logarithmic term

convective heat transfer in constant porosity media was first examined by Vafai and Tien [3], while a complete and detailed analysis of these effects at the interface region of a porous medium was performed by Vafai and Thiyagaraja [4]. It was shown that both effects decrease the velocity of the streaming fluid in the boundary layer and thereby reduce the rate of heat transfer. In some applications, such as packed-sphere beds and chemical catalytic reactors, the porosity of the porous medium is no longer a constant. For packed spheres Benenati and Brasilow [5] have shown that the porosity near the solid surface is larger than that in the main stream. This nonuniform porosity distribution causes the so-called channelling phenomenon first studied by Vortmeyer and Schuster [6] who found that the velocity attains a maximum very close to the solid boundary. It was observed that the flow channelling effect has a significant influence on convective flow, i.e. it promotes momentum transport in the boundary-layer and this causes an enhanced rate of heat transfer.

Some investigators, notably Chandrasekhara and Vortmeyer [7], Vafai [8], Chandrasekhara et al. [9,10] and Cheng et al. [11–13], have incorporated a variable permeability in the flow past and through a porous medium and have also shown that the velocity distribution and heat transfer are affected greatly. In studying nonuniform permeability effects a simple exponential function of the distance from the wall was usually employed. For the convective boundary-layer problems this function usually leads to nonsimilar boundary-layer equations, but Chandrasekhara et al. [9,10] enforced self similarity by allowing the region of increased permeability to grow in size in the same way as the boundary-layer itself.

The aim of the present paper is to study how an exponentially decaying permeability affects the free convective boundary-layer flow induced by a vertical heated surface embedded in a porous medium and held at a constant temperature. The governing nonsimilar boundary-layer equations are solved by using the Keller box method for various values of the permeability parameter γ which measures the ratio of the permeability at the wall to that at infinity. The numerical results are compared with an asymptotic expansion valid for relatively large distances from the leading edge. In this asymptotic regime the uniform permeability solution is valid at leading order, but the flow and heat transfer are most strongly affected nearer the leading edge where the growing boundary-layer is still embedded within the region where the permeability varies strongly.

In Section 2, we derive the boundary layer equations setting them into the context of the constant permeability problem first studied by Cheng and Minkowycz [14]. The boundary layer flow both near to and far from the leading edge are studied using asymptotic methods in Section 3. Finally, the numerical results are presented and these together with the asymptotic results are discussed in Section 4.

2. Basic equations

Consider a vertical semi-infinite flat plate with constant surface temperature, T_w, which is embedded in a fluid-saturated porous medium of variable permeability and of otherwise uniform temperature, T_∞, where $T_w > T_\infty$. The coordinate system is such that the origin is placed at the leading edge and that the x^*-axis is measured along the surface in the upward vertical direction; the y^*-axis is measured perpendicularly from the surface into the porous medium. It is assumed here that the permeability, $K(y^*)$, of the porous medium varies as

$$K(y^*) = K_\infty + (K_w - K_\infty)e^{-y^*/d}, \tag{1}$$

where K_w is the permeability at the wall, K_∞ is the permeability of the ambient medium, and d is the length scale over which the permeability varies.

After suitable nondimensionalisation the steady two-dimensional equations which govern the flow may be written as

$$u_x + v_y = 0, \tag{2a}$$

$$u = -A(y)\left[p_x - Ra\theta\right], \tag{2b}$$

$$v = -A(y)p_y \tag{2c}$$

$$u\theta_x + v\theta_y = \theta_{xx} + \theta_{yy}, \tag{2d}$$

where u and v are the velocity components along the x- and y-axes, respectively, p is the pressure, θ is the temperature, Ra is the Darcy–Rayleigh number based on d and K_∞, and $A(y)$ is the nondimensional permeability given by

$$A(y) = 1 + (\gamma - 1)e^{-y}, \tag{3}$$

with $\gamma = K_w/K_\infty$ being the permeability parameter. Given how naturally occurring and artificial media are packed near solid boundaries we restrict attention to values of γ which are greater than 1.

When $\gamma = 1$, which corresponds to a constant permeability, and when Ra is very large, the resulting boundary-layer flow is self-similar. But the presence of a nonuniform permeability field renders the boundary-layer nonsimilar. Thus, if we define a streamfunction, ψ, in the usual way according to $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, then the boundary-layer equations may be obtained by using the scalings,

$$y = \bar{y}, \quad x = Ra\,\bar{x}, \quad \psi = Ra\,\bar{\psi}. \tag{4}$$

Substitution of Eq. (4) into Eqs. (2) and allowing Ra to become asymptotically large gives the following equations at leading order

$$\bar{\psi}_{\bar{y}} = A(\bar{y})\theta, \tag{5a}$$

$$\theta_{\overline{yy}} = \bar{\psi}_{\bar{y}}\theta_{\bar{x}} - \bar{\psi}_{\bar{x}}\theta_{\bar{y}} \tag{5b}$$

which have to be solved subject to the boundary conditions,

$$\bar{\psi} = 0, \quad \theta = 1 \text{ on } \bar{y} = 0 \quad \text{and } \theta \to 0 \quad \text{as } \bar{y} \to \infty. \tag{5c}$$

We note that the scalings used in Eq. (4) correspond to the situation in which the boundary-layer thickness is of the same order of magnitude as the lengthscale of the permeability variation and therefore $y = O(1)$ is set in this paper, rather than $x = O(1)$ which is most often used when there is no natural lengthscale in the flow.

A boundary-layer transformation may now be introduced which is very closely related to that of Cheng and Minkowycz [14] who were the first to consider the constant-permeability counterpart of the present problem. Therefore we set

$$\bar{\psi} = \xi f(\xi, \eta) \tag{6a}$$

$$\theta = g(\xi, \eta) \tag{6b}$$

where

$$\xi = \bar{x}^{1/2} \quad \text{and} \quad \eta = \bar{y}/\bar{x}^{1/2}. \tag{7}$$

Therefore f and g satisfy the equations

$$f' = \left[(\gamma - 1)e^{-\xi\eta} + 1\right]g,, \tag{8a}$$

$$g'' + \frac{1}{2}fg' = \frac{1}{2}\xi\left(f'g_\xi - g'f_\xi\right) \tag{8b}$$

where primes denote partial derivatives with respect to η, and the boundary conditions become

$$f = 0, \quad g = 1 \quad \text{at } \eta = 0, \quad \text{and} \quad g(\eta) \to 0 \text{ as } \eta \to \infty. \tag{9}$$

Finally, the nondimensional rate of heat transfer at the plate can be determined from the expression

$$Q = \frac{\partial g}{\partial \eta}\Big|_{\eta=0}. \tag{10}$$

3. Asymptotic analyses

3.1. Analysis near the leading edge

In this subsection we will present the leading order term in a small-ξ expansion; further terms could be presented but are omitted since their purpose would only be to provide further verification of the accuracy of the numerical code and they would not add any physical insight.

When $\xi = 0$ Eqs. (8) become

$$f' = \gamma g, \tag{11a}$$

$$g'' + \frac{1}{2} f g' = 0 \tag{11b}$$

subject to Eq. (9). If we make the transformation

$$\eta = \gamma^{-1/2}\hat{\eta}, \qquad f(\eta) = \gamma^{1/2}\hat{f}(\hat{\eta}), \tag{12}$$

then Eq. (11) reduce to the classical form of Cheng and Minkowycz [14], namely

$$\hat{f}' = \hat{g}, \tag{13a}$$

$$\hat{g}'' + \frac{1}{2}\hat{f}\hat{g}' = 0 \tag{13b}$$

subject to

$$\hat{f} = 0, \hat{g} = 1 \quad \text{at}\, \hat{\eta} = 0, \quad \text{and} \qquad \hat{g}(\hat{\eta}) \to 0 \text{ as } \hat{\eta} \to \infty, \tag{13c}$$

where primes on $\hat{f}$ and $\hat{g}$ denote derivatives with respect to $\hat{\eta}$. A consequence of this transformation is that we have

$$g'(0) = \gamma^{1/2}\hat{g}'(0) = -0.44375\gamma^{1/2} \tag{14}$$

where the value of $\hat{g}'(0)$ has been taken from Rees and Bassom [15] and Rees and Pop [16]. Clearly, as γ rises from the value of 1, the uniform permeability case, the fluid is able to advect more easily up the heated surface, thereby thinning the boundary layer and increasing the rate of heat transfer.

3.2. Analysis for large values of ξ

At large distances from the leading edge the region where the permeability of the medium varies becomes confined to a near-wall region of decreasing thickness compared with that of the boundary-layer itself which grows as $x^{1/2}$. In terms of y this region is of constant thickness, and we would expect that the boundary-layer splits into two asymptotic regions: a main region where $\eta = O(1)$, wherein the flow looks very much like the self-similar flow in a medium of uniform permeability, and a near-wall region where $y = O(1)$ within which variable permeability effects dominate. The following analysis is undertaken using the method of matched asymptotic expansions.

In the main region the solutions may be expanded in the form:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f_0(\eta) \\ g_0(\eta) \end{pmatrix} + \xi^{-1}\begin{pmatrix} f_1(\eta) \\ g_1(\eta) \end{pmatrix} + \xi^{2}\ln\xi\begin{pmatrix} \tilde{f}_2(\eta) \\ \tilde{g}_2(\eta) \end{pmatrix} + \xi^{-2}\begin{pmatrix} f_2(\eta) \\ g_2(\eta) \end{pmatrix} + \cdots, \tag{15a}$$

while in the near-wall region we use

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F_0(y) \\ G_0(y) \end{pmatrix} + \xi^{-1}\begin{pmatrix} F_1(y) \\ G_1(y) \end{pmatrix} + \xi^{-2}\begin{pmatrix} F_2(y) \\ G_2(y) \end{pmatrix} + \xi^{3}\ln\xi\begin{pmatrix} \tilde{F}_3(y) \\ \tilde{G}_3(y) \end{pmatrix} + \xi^{-3}\begin{pmatrix} F_3(y) \\ G_3(y) \end{pmatrix} + \cdots, \tag{15b}$$

where the logarithmic terms are the eigensolutions corresponding to the leading-edge shift effect, see Refs. [17] and Daniels and Simpkins [18]. The details of the solution procedure are straightforward since they closely follow the procedure used in Ref. [18], and therefore we omit much of the analysis.

The functions f_0 and g_0 satisfy Eq. (13) and thus,

$$g_0'(0) = -0.44375 \equiv -a_0. \tag{16}$$

At $O(\xi^{-1})$ we easily find that

$$f_1 = 1 - \gamma, \qquad g_1 = 0. \tag{17}$$

At $O(\xi^{-2}\ln\xi)$ the eigenfunctions $\tilde{f}_2$ and $\tilde{g}_2$ are given by

$$\tilde{f}_2 = \lambda(\eta f_0' - f_0), \qquad \tilde{g}_2 = \lambda\eta g_0' \tag{18}$$

where λ is an eigenvalue whose numerical value is determined by insisting that the equations at $O(\xi^{-2})$ have a solution. At $O(\xi^{-2})$ the equations are

$$f_2' = g_2, \tag{19a}$$

$$g_2'' + \frac{1}{2}(f_0 g_2' + 2f_0' g_2 - f_2 g_0') = \frac{1}{2}\lambda f_0 f_0'' \tag{19b}$$

subject to the boundary and matching conditions,

$$f_2(0) = -a_0(\gamma - 1), \quad g_2(0) = 0, \quad g_2(\eta \to \infty) \to 0. \tag{19c}$$

Although Eq. (19) determine the value of λ, arbitrary multiples of the eigensolution given in Eq. (18) are also involved in the solution for f_2 and g_2, and therefore we cannot be precise about the form these functions take. The value of λ may, nevertheless, be evaluated analytically. If we multiply Eq. (19b), written in terms of f_2 using Eq. (19a), by f_0 and integrate between $\eta = 0$ and $\eta = \infty$, then we obtain

$$\lambda = 2(\gamma - 1)a_0^2. \tag{20}$$

This formula shows that there is no logarithmic term when the medium has uniform permeability.

In the near-wall region we find the following solutions,

$$F_0 = 0, \qquad G_0 = 1 \tag{21a}$$

$$F_1 = (1-\gamma)(1-\mathrm{e}^{-y}) + y, \qquad G_1 = -a_0 y, \tag{21b}$$

$$F_2 = a_0(\gamma - 1)\left(y\mathrm{e}^{-y} + \mathrm{e}^{-y} - 1\right) - \frac{1}{2}a_0 y^2, \qquad G_2 = 0 \tag{21c}$$

For the purposes of obtaining the first two terms in the large-ξ expansion for the surface rate of heat transfer we need to proceed to $O(\xi^{-3}\ln\xi)$ in the near-wall region. We find that

$$\tilde{G}_3(0) = \left[\tilde{g}_2'(0)\right]y = -a_0\lambda y. \tag{22}$$

Given that

$$Q = \frac{\partial g}{\partial \eta}\Big|_{\eta=0} = \xi\left(\frac{\partial G_0}{\partial y} + \xi^{-1}\frac{\partial G_1}{\partial y} + \xi^{-2}\frac{\partial G_2}{\partial y} + \xi^{-3}\ln\xi\,\frac{\partial \tilde{G}_3}{\partial y} + \cdots\right)_{y=0}, \tag{23}$$

we obtain

$$Q = -a_0\left[1 + 2a_0^2(\gamma - 1)\xi^{-2}\ln\xi\right] + O\left(\xi^{-2}\right). \tag{24}$$

4. Results and discussion

Eqs. (8) subject to Eq. (9) were solved numerically using the Keller box method. In the ξ direction 401 equally spaced points were taken to lie in the range $0 \le \xi \le 20$, while a nonuniform grid of 78 points was taken in the η direction. Gridpoints were concentrated towards the $\eta = 0$ boundary in order to capture well the relatively thin boundary layer which occurs there. A value of $\eta_{\max} = 30$ was taken, as this lies well outside the boundary layer in all circumstances.

As γ increases the local rate of heat transfer, as

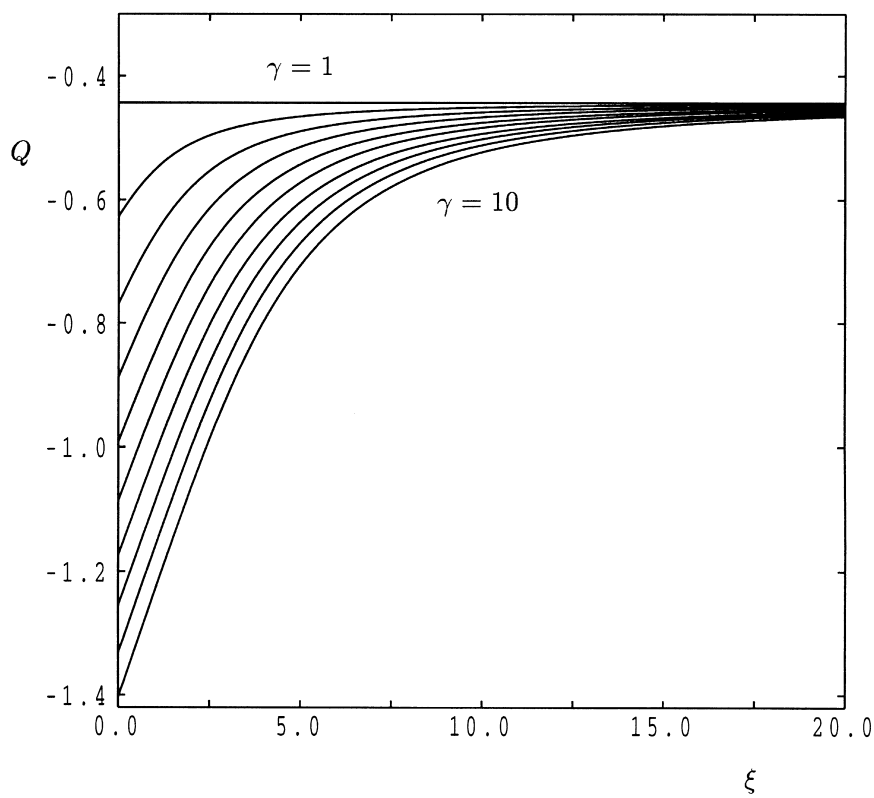

Fig. 1. Variation of the local rate of heat transfer, Q, with ξ for different values of γ. The straight line corresponds to $\gamma = 1$, the uniform permeability case. The other curves are for $\gamma = 2, 3, \ldots, 10$.

shown in Fig. 1, increases in magnitude at any chosen value of ξ; this is because the increased near-wall permeability allows the fluid to advect heat away more quickly than it would for $\gamma = 1$, thereby thinning the boundary layer and increasing the temperature gradient. For any chosen value of $\gamma \neq 1$ the rate of heat transfer varies monotonically from the value given by Eq. (14) towards that given by the leading order term in Eq. (22), which corresponds to the uniform permeability result of Cheng and Minkowycz [14]. The decay towards the common asymptotic value, $-a_0$, is approximately proportional to ξ^{-2}, but since there are unknown contributions of $O(\xi^{-2})$ compared with the known $O(\xi^{-2} \ln \xi)$ values, it is impossible to distinguish between these terms given such small values of ξ as were used here. Therefore, a strong quantitative confirmation of the validity of the large-ξ asymptotic expansion by a cross-validation using the numerical results is not feasible in this case.

Typical profiles of the temperature, g, are shown in Fig. 2. Here, $\gamma = 10$ was chosen as this case gives a very strong channelling effect near the heated surface. As ξ increases, the approach to the self-similar profile is very evident in Figs. 1 and 2, and the strong channelling effect at small values of ξ corresponds to large values of f as $\eta \rightarrow \infty$, and large gradients in g at the heated surface.

The qualitative nature of the present results are consistent with the analogous problem of free convection in a layered porous medium. Although Rees [19] considered a variety of layered configurations, the most relevant one for the present paper consists of a uniform thickness layer of uniform permeability sandwiched between a uniform temperature heated vertical surface and an ambient porous medium with a different permeability; this may be considered as a discrete version of the present configuration. Channelling effects were also observed and are again most pronounced at stations close to the leading edge.

Two other recent papers have dealt with certain extensions of Darcy's law which result in a twin-layer structure of the boundary-layer at large distances from the leading edge. These are: (i) boundary effects as modelled by the Brinkman terms [20], (ii) local thermal nonequilibrium between the solid and fluid phases of the porous medium [21]. In all three cases (the present paper being the third) the Cheng and Minkowycz solution is obtained at leading order in the main part of the boundary-layer, but the flows described in detail in Refs. [20,21] also have a twin-layer structure near the

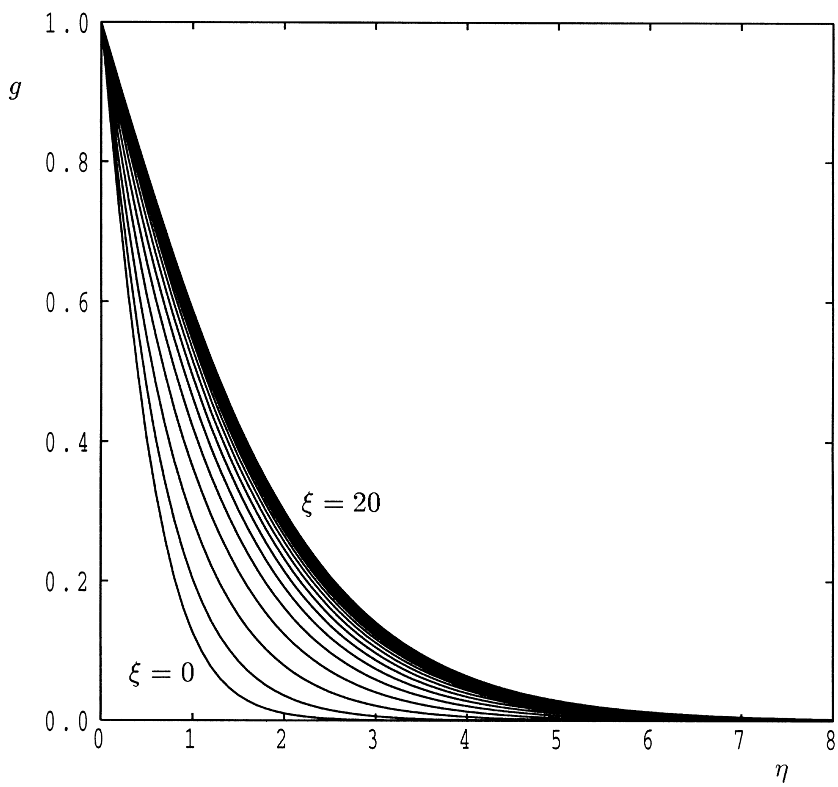

Fig. 2. Temperature profiles for various values of ξ for the case $\gamma = 10$. The lowest curve corresponds to $\xi = 0$. Successive curves correspond to $\xi = 1, 2, \ldots, 20$.

leading edge. It is clear in all three problems that these variations upon a theme' serve to modify the flow and heat transfer characteristics quite substantially in a very large portion of the porous medium. It would now be of some substantial interest to investigate how these several effects interact with one another.

References

[1] D.A. Nield, A. Bejan, Convection in Porous Media, 2nd ed., Springer, New York, 1998.

[2] D.B. Ingham, I. Pop (Eds.), Transport Phenomena in Porous Media, Pergamon Press, Oxford, 1998.

[3] K. Vafai, C.L. Tien, Boundary and inertia effects on flow and heat transfer in porous media, Int. J. Heat Mass Transfer 24 (1981) 195–203.

[4] K. Vafai, R. Thiyagaraja, Analysis of the flow and heat transfer at the interface region of a porous medium, Int. J. Heat Mass Transfer 30 (1987) 1391–1405.

[5] R.F. Benenati, C.B. Brasilow, Void fraction distribution in packed beds, A.I.Ch.Eng. J. 8 (1962) 359–361.

[6] D. Vortmeyer, J. Schuster, Evaluation of steady flow profiles in rectangular and circular packed beds by a variational method, Chem. Eng. Sci. 38 (1983) 1691–1699.

[7] B.C. Chandrasekhara, D. Vortmeyer, Flow model for velocity distribution in fixed porous beds under isothermal conditions, Wärme and Stoffübertr 12 (1979) 105–111.

[8] K. Vafai, Convective flow and heat transfer in variable porosity media, J. Fluid Mech 147 (1984) 233–259.

[9] B.C. Chandrasekhara, P.M.S. Hamboodiri, A.R. Hanumanthappa, Similarity solutions for buoyancy-induced flows in a saturated porous medium adjacent to impermeable horizontal surfaces, Wärme and Stoffübertr 18 (1984) 17–23.

[10] B.C. Chandrasekhara, P.M.S. Hamboodiri, Influence of variable permeability on combined free and forced convection about inclined surfaces in porous media, Int. J. Heat Mass Transfer 28 (1985) 199–206.

[11] P. Cheng, C.T. Hsu, Fully developed forced convective flow through an annular packed-sphere bed with wall effects, Int. J. Heat Mass Transfer 29 (1986) 1843–1853.

[12] P. Cheng, H. Zhu, Effects of radial thermal dispersion on fully developed forced convection in cylindrical packed tubes, Int. J. Heat Mass Transfer 30 (1987) 2373–2383.

[13] P. Cheng, D. Vortmeyer, Transverse thermal dispersion and wall channelling in a packed bed with forced convection flow, Chem. Eng. Sci 43 (1988) 2523–2532.

[14] P. Cheng, W.J. Minkowycz, Free convection about a vertical flat plate imbedded in a porous medium with application to heat transfer from a dike, J. Geophys. Res 82 (1977) 2040–2044.

[15] D.A.S. Rees, A.P. Bassom, Some exact solutions for free convective flows over semi-infinite surfaces in porous media, Int. J. Heat Mass Transfer 34 (1991) 156–1567.

[16] D.A.S. Rees, I. Pop, A note on free convection along a vertical wavy surface in a porous medium, Trans. A.S.M.E. J. Heat Transfer 116 (1994) 505–508.

[17] K. Stewartson, On asymptotic expansions in the theory of boundary layers, J. Math. Phys. 36 (1957) 173–191.

[18] P.G. Daniels, P.G. Simpkins, The flow induced by a heated vertical wall in a porous medium, Quart. J. Mech. appl. Math. 37 (1984) 339–354.

[19] D.A.S. Rees, Free convective boundary layer flow from a heated surface in a layered porous medium, Journal of Porous Media 2 (1999) 39–58.

[20] D.A.S. Rees, K. Vafai, Darcy–Brinkman free convection from a heated horizontal surface, Numerical Heat Transfer: Part A — Applications 35 (1999) 191–204.

[21] D.A.S. Rees, I. Pop, Vertical free convective boundary-layer flow in a porous medium using a thermal nonequilibrium model, Journal of Porous Media, in press.